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Exact transient solutions to nonlinear diffusion–convection equations in higher dimensions

M P Edwards and P Broadbridge
Department of Mathematics, University of Wollongong, Wollongong, NSW 2522, Australia

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Abstract. The complete Lie algebra of classical infinitesimal symmetries of the nonlinear diffusion–convection equation in two and three dimensions is presented. Except for some cases involving constant diffusivity, a complete reduction to an ordinary differential equation is not possible. However, closed-form solutions are obtained for special forms of both the 2D and 3D nonlinear diffusion–convection equations, using a symmetry reduction and an additional physical constraint. This extends the small list of closed-form transient solutions already known.

1. Introduction

The nonlinear diffusion–convection equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(u)\nabla u) - \frac{\mathrm{d}K}{\mathrm{d}u}\frac{\partial u}{\partial z} \tag{1}$$

with ∇ the Laplacian operator in N spatial dimensions ($N = 1, 2$ or 3) has a variety of applications to porous media, including displacement of one liquid by another (Fokas and Yortsos 1982), unsaturated flow (Klute 1952), transport of a solute with adsorption to pore surfaces (Rosen 1982), and saturated flow in a swelling medium (Smiles and Rosenthal 1968). $D(u)$ is the concentration-dependent diffusivity and, based on Darcys' law (Klute 1952) for hydrological flows, K will be viewed as the concentration-dependent conductivity.

Because of its practical importance, much work has been devoted to constructing exact and approximate solutions to (1). Time-dependent solutions have been found mainly in one spatial dimension. These solutions are of two types. The first type relies on special integrable models which can be transformed to the linear diffusion equation. The integrable models are the Burgers' equation (Clothier *et al* 1981) (where D is constant and K is quadratic), and the Fokas–Yortsos–Rosen equation (Fokas and Yortsos 1982, Rosen 1982) (where $D(u) = a/(b-u)^2$, $K'(u) = \lambda/2(b-u)^2$ with a, b and λ constant). In the second method of solution, similarity solutions follow from classical Lie symmetry group reductions when $D(u)$ is a power law or an exponential (Oron and Rosenau 1986).

For higher dimensions, much less is known. Unlike the one-dimensional case there are no linearizable models in two dimensions (Broadbridge 1986) and we doubt that integrable models exist in three dimensions. Next we turn to the other best known approach for obtaining exact solutions. In section 2 classical Lie group symmetries are investigated for the general class of equations (1) in two dimensions. We find that the only forms which have additional symmetries, beyond translations and rotations, are with power-law, log and exponential functions for $D(u)$ and $K(u)$. Full reduction to an ordinary differential

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equation (ODE) is possible in only one case, the 2D Burgers' equation, i.e. D constant and $K(u)$ quadratic.

Recently, Philip and Knight (1991) have obtained a two-step reduction to an ODE for power-law forms of $D(u)$ and $K(u)$ where (1) is expressed in polar coordinates. Solutions obtained in this way follow neither from integrable models nor from two-stage classical symmetry reductions. We show that only the first stage of this technique is the result of a classical group reduction. In fact, the reduced equation after the first step can be shown to have no classical symmetries. The reduction used by Philip and Knight has a power-law time dependence. However, their solution method does not apply to the case $D = u^{-1}$ in two dimensions or to the case $D = u^{-2/3}$ in three dimensions. In sections 3 and 4 we extend the method of Philip and Knight to these singular cases. We find that for these special cases, we obtain a much wider set of solutions than are available in any other single model.

2. Lie group symmetry analysis

We consider the classical Lie group symmetry analysis of the class of equations (1) in two dimensions. However, we are no longer specifically concerned with unsaturated flow in a porous medium, but in the general equation, so that $D(u)$ and $K(u)$ are completely arbitrary. We then identify any special forms of D and K which possess additional symmetries.

We consider the infinitesimal transformation

$$\begin{aligned} u_* &= e^{\epsilon\Gamma}u = u + \epsilon\,\mathcal{U}(x,z,t,u) + O(\epsilon^2) \qquad & t_* &= e^{\epsilon\Gamma}t = t + \epsilon\,\mathcal{T}(x,z,t,u) + O(\epsilon^2) \\ x_* &= e^{\epsilon\Gamma}x = x + \epsilon\,\mathcal{X}(x,z,t,u) + O(\epsilon^2) \qquad & z_* &= e^{\epsilon\Gamma}z = z + \epsilon\,\mathcal{Z}(x,z,t,u) + O(\epsilon^2) \end{aligned} \tag{2}$$

where

$$\Gamma = \mathcal{X}\frac{\partial}{\partial x} + \mathcal{Z}\frac{\partial}{\partial z} + \mathcal{T}\frac{\partial}{\partial t} + \mathcal{U}\frac{\partial}{\partial u}$$

is the infinitesimal generator (Olver 1986, Bluman and Kumei 1989). We then extend (2) to first and second order by the prolongation formulae, for example,

$$\frac{\partial u_*}{\partial x_*} = \frac{\partial u}{\partial x} + \epsilon\,\mathcal{U}_1 + O(\epsilon^2)$$

where

$$\mathcal{U}_1 = \frac{D}{Dx}\mathcal{U} - \left(\frac{D}{Dx}\mathcal{X}\right)u_x - \left(\frac{D}{Dx}\mathcal{Z}\right)u_z - \left(\frac{D}{Dx}\mathcal{T}\right)u_t \tag{3}$$

and D/Dx is the total derivative operator with respect to x

$$\frac{D}{Dx}F(x,z,t,u) = \frac{\partial F}{\partial x} + u_x\frac{\partial F}{\partial u}\,.$$

Invariance of the governing equation (1) under the infinitesimal transformation (2) and assuming that the derivatives of u are independent leads to a set of determining relations, which are linear partial differential equations (PDEs) in $\mathcal{X}, \mathcal{Z}, \mathcal{T}$ and $\mathcal{U}$. The symmetry analysis was performed using the software package Dimsym under Reduce (Sherring 1993).

For totally arbitrary D and K, the only symmetries are the generators of the space and time translations

$$\Gamma_1 = \frac{\partial}{\partial x} \qquad \Gamma_2 = \frac{\partial}{\partial z} \qquad \Gamma_3 = \frac{\partial}{\partial t}\,.$$

Up to a linear change of variables, the only functional forms of $D(u)$ which have extra symmetries are power law and exponential. In table 1, Γ_1, Γ_2 and Γ_3 are not listed, as they

Table 1. Symmetries for the 2D diffusion–convection equation $m, n \in \Re$.

$D(u)$	$K'(u)$	Γ_i
u^m	u^n	$\Gamma_4 = (m-2n)t\frac{\partial}{\partial t} + (m-n)x\frac{\partial}{\partial x} + (m-n)z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}$
u^m	$\ln(u)$	$\Gamma_4 = mt\frac{\partial}{\partial t} + mx\frac{\partial}{\partial x} + (mz+t)\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}$
e^{mu}	e^{nu}	$\Gamma_4 = (m-2n)t\frac{\partial}{\partial t} + (m-n)x\frac{\partial}{\partial x} + (m-n)z\frac{\partial}{\partial z} + \frac{\partial}{\partial u}$
e^{mu}	u	$\Gamma_4 = mt\frac{\partial}{\partial t} + mx\frac{\partial}{\partial x} + (mz+t)\frac{\partial}{\partial z} + \frac{\partial}{\partial u}$
const	u	$\Gamma_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}$
		$\Gamma_5 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial u}$

are common to all cases. We do not list the case $K'(u)$ constant as this can be transformed to the case of pure diffusion, analysed fully by Galaktionov *et al* (1986).

Unlike the one-dimensional Burgers' equation, the two-dimensional Burgers' equation (D constant, K quadratic) cannot, in general be transformed to a linear equation (Broadbridge 1986). However, the two-dimensional Burgers' equation does have special symmetry properties. It is the only two-dimensional form of (1) that has a fifth symmetry. The two non-trivial symmetries Γ_4 and Γ_5 are compatible because they obey the simple commutation property $[\Gamma_4, \Gamma_5] = \Gamma_5$. Hence, the two-dimensional Burgers' equation is the only two-dimensional nonlinear convection–diffusion equation that can be fully reduced to an ODE by classical Lie symmetry reductions.

3. Exact solutions for a 2D nonlinear diffusion–convection equation

We transform (1) from Cartesian space coordinates (x, z) into cylindrical polar coordinates (r, γ) by

$$x = r\sin\gamma \qquad z = r\cos\gamma$$

to obtain the nonlinear PDE

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD(u)\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \gamma}\left[D(u)\frac{\partial u}{\partial \gamma}\right] - K'(u)\left[\cos\gamma\frac{\partial u}{\partial r} - \frac{\sin\gamma}{r}\frac{\partial u}{\partial \gamma}\right]. \tag{4}$$

If we assume power-law functions for $D(u)$ and $K'(u)$, (i.e. $D(u) = u^m$, $K'(u) = u^n$), then the appropriate symmetries are

$$\begin{aligned} &\Gamma_1 = \sin\gamma\frac{\partial}{\partial r} + \frac{\cos\gamma}{r}\frac{\partial}{\partial \gamma} \qquad \Gamma_2 = \cos\gamma\frac{\partial}{\partial r} - \frac{\sin\gamma}{r}\frac{\partial}{\partial \gamma} \\ &\Gamma_3 = \frac{\partial}{\partial t} \qquad \Gamma_4 = (m-2n)t\frac{\partial}{\partial t} + (m-n)r\frac{\partial}{\partial r} + u\frac{\partial}{\partial u}. \end{aligned} \tag{5}$$

For total material conservation, Philip and Knight (1991) assume the ad hoc functional form

$$u = F(\rho, \gamma)t^{-\alpha} \qquad \rho = rt^{-\alpha/2} \tag{6}$$

and show that the values of α and n must be

$$\alpha = \frac{1}{m+1} \qquad n = m + \tfrac{1}{2} \qquad m \neq -1.$$

If we consider the symmetries (5), this gives

$$\Gamma_4 = 2(m+1)t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - 2u\frac{\partial}{\partial u}. \tag{7}$$

From this symmetry we obtain the characteristic equation

$$\frac{\mathrm{d}r}{r} = \frac{\mathrm{d}\gamma}{0} = \frac{\mathrm{d}t}{2(m+1)t} = \frac{\mathrm{d}u}{-2u}$$

which may be solved to yield the functional form

$$u = F(\rho, \gamma)t^{-1/(m+1)} \qquad \rho = rt^{-1/2(m+1)} \qquad m \neq -1 \tag{8}$$

which is precisely the form used by Philip and Knight (1991) to obtain a reduction of variables. The special case $m = -1$ is excluded from this reduction. The ad hoc functional form assumed by Philip and Knight (1991) is incapable of treating the case $m = -1$. Our systematic symmetry approach identifies the correct invariant variable substitutions not only for their cases but also for this exceptional case. This shows that a systematic symmetry analysis obviates the need for a case by case search for appropriate variable substitutions.

From the work of Philip and Knight (1991) which makes no reference to symmetry analysis, it is not clear how to treat the special case $m = -1$. This inverse linear case arises specifically in the analysis of a diffusing electron cloud in thermal equilibrium (Lonngren and Hirose 1976). In the hydrology context, it has two applications. Firstly, in rigid field soils containing biomacropores, the soil water diffusivity may be weakly increasing (Clothier and White 1981). Such a diffusivity may well be represented by $D(\theta) = a(b - \theta)^{-1}$ with b greater than the porosity. Secondly, in a saturated swelling paste, the effective material diffusivity may be a weakly decreasing function which could be represented by $D_m(\vartheta) = a(b + \vartheta)^{-1}$ (Broadbridge 1990).

For the case $m = -1$, the symmetry (7) used above may be replaced by

$$\Gamma_4 = r\frac{\partial}{\partial r} - 2u\frac{\partial}{\partial u}\,.$$

If we now consider a linear combination of Γ_3 and Γ_4, then for $m = -1$, the characteristic equations take the form

$$\frac{\mathrm{d}r}{r} = \frac{\mathrm{d}\gamma}{0} = \frac{\mathrm{d}t}{\beta} = \frac{\mathrm{d}u}{-2u}$$

where $\beta \in \Re$, $\beta \neq 0$. These may be solved to obtain

$$u = F(\rho, \gamma)\mathrm{e}^{-2t/\beta} \qquad \rho = r\mathrm{e}^{-t/\beta}\,.$$

If we let $c = 2/\beta$, then

$$u = F(\rho, \gamma)\mathrm{e}^{-ct} \qquad \rho = r\mathrm{e}^{-ct/2}\,. \tag{9}$$

This will lead to a reduction by one of the number of independent variables of the PDE (4). This form of u also ensures total conservation of material, so for this case $m = -1$, we can also seek a solution of (4) by extending the method of Philip and Knight (1991).

In fact, for this case, the functional form involves an arbitrary constant c, which will mean greater variability in the solution. We also note that, unlike the functional form (6) with power-law dependence on t, the new solution (9) has the additional advantage that it is finite for $t = 0$. It is worthwhile noting that although a functional form

$$u = F(\rho, \gamma)\mathrm{e}^{-t/\beta} \qquad \rho = r\mathrm{e}^{nt/\beta}$$

is valid in general when $m = 2n$, it is only when $n = -\frac{1}{2}$, i.e. when $m = -1$, that the global material conservation condition is satisfied.

This condition requires zero flux at the origin. That is

$$\text{for} \quad t > 0 \qquad r = 0 \qquad \frac{\partial u}{\partial r} = 2\cos\gamma\, u^{3/2}\,. \tag{10}$$

Substitution of (9) into (4) and (10) gives

$$\frac{-c}{2\rho}\frac{\partial}{\partial\rho}\left[\rho^2 F\right] = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho F^{-1}\frac{\partial F}{\partial\rho}\right] + \frac{1}{\rho^2}\frac{\partial}{\partial\gamma}\left[F^{-1}\frac{\partial F}{\partial\gamma}\right] - F^{-1/2}\left[\cos\gamma\frac{\partial F}{\partial\rho} - \frac{\sin\gamma}{\rho}\frac{\partial F}{\partial\gamma}\right] \tag{11}$$

with the zero flux condition becoming

$$t > 0 \qquad \rho = 0 \qquad \frac{\partial F}{\partial\rho} = 2\cos\gamma\ F^{3/2}. \tag{12}$$

As in the case of similarity variables with power-law time dependence (Philip and Knight 1991), the similarity reduction (9) implies that the flux has no component normal to the radii. This provides a constraint

$$\frac{-1}{\rho}F^{-1}\frac{\partial F}{\partial\gamma} - 2F^{1/2}\sin\gamma = 0 \tag{13}$$

which allows us to make a further reduction of variables, even though classical symmetry provides no alternative additional constraint. For our purposes, the only significance of global mass conservation is that it provides an extra restriction (13) which allows us to make a further reduction of variables. Without this extra restriction of global mass conservation, a further reduction of variables could be made possible by an additional symmetry. However we have found (table 1) that the extra symmetry exists only for the case of the 2D Burgers' equation. Of course, each diffusion–convection equation (1), being a conservation equation, locally conserves material, even if it is unbalanced globally due to boundary conditions. However, it is only the global mass conserving solutions that allow this further reduction.

Taking $(-1/\rho)(\partial/\partial\gamma)$ of (13) and substitution of this expression into (11) leads to

$$\frac{-c}{2\rho}\frac{\partial}{\partial\rho}\left[\rho^2 F\right] = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho F^{-1}\frac{\partial F}{\partial\rho}\right] - \frac{2\cos\gamma}{\rho}\frac{\partial}{\partial\rho}\left[\rho F^{1/2}\right] \tag{14}$$

an ODE for each value of γ. This may be integrated with respect to ρ, with the constant of integration eliminated through use of (12). Thus we wish to solve the first-order ODE

$$F^{-2}\frac{\partial F}{\partial\rho} = 2\cos\gamma F^{-1/2} - \tfrac{1}{2}c\rho \tag{15}$$

subject to the condition

$$\rho = 0 \qquad F = F_0 > 0\,.$$

If we let $\eta = F^{-1}$ *then (15) becomes*

$$\frac{\partial\eta}{\partial\rho} = -2\cos\gamma\,\eta^{1/2} + \tfrac{1}{2}c\rho \tag{16}$$

subject to $\rho = 0, \eta = \eta_0 = F_0^{-1}$.

The ODE (16) may be solved exactly (Kamke 1959) to give the solution (for $c > 0$)

$$\left(\frac{\eta + \rho\eta^{1/2}\cos\gamma - \frac{1}{4}c\rho^2}{\eta_0}\right) = \left(\frac{2\eta^{1/2} + \rho(\cos\gamma - \sqrt{\cos^2\gamma + c})}{2\eta^{1/2} + \rho(\cos\gamma + \sqrt{\cos^2\gamma + c})}\right)^{2\cos\gamma/\sqrt{\cos^2\gamma+c}}\,. \tag{17}$$

The case $c < 0$ implies a physically unappealing solution which increases exponentially in t. For such a globally mass-conserving solution to a dissipative equation, this backward evolution cannot occur if the solution is smooth. For example, from the analogous solution to (17) but with $c < 0$, we find that along the ray $\gamma = \pi/2$, the solution is given by

$$u = \frac{e^{-ct}}{[\eta_0 + c\rho^2/4]} \tag{18}$$

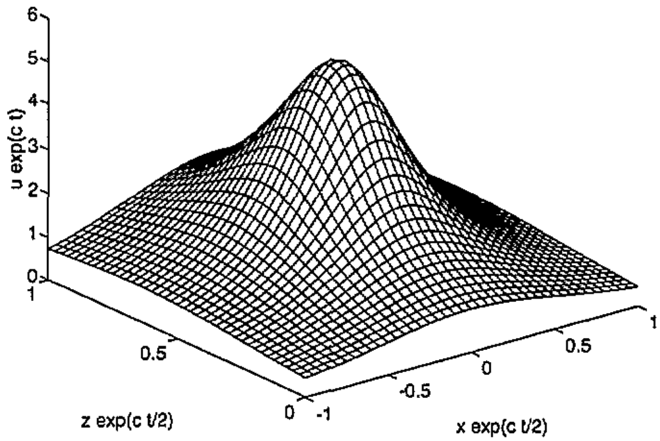

Figure 1. Plot of (17) for $c = 5.0$ and $F_0 = 1.0$.

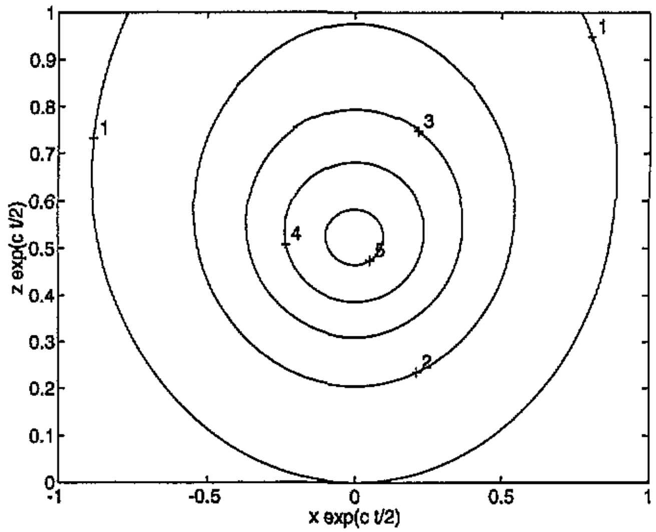

Figure 2. Contour plot of (17) for $c = 5.0$ and $F_0 = 1.0$.

which is singular at $\rho = 2\sqrt{\eta_0/|c|}$.

Previously, the reduction method of Philip and Knight (1991) has produced closed-form two-dimensional solutions only in the cases $(m, n) = (0, \frac{1}{2})$ and $(m, n) = (\frac{1}{2}, 1)$. Here, we have contributed an infinite family of new solutions whose character changes as the parameter c varies.

In figures 1 and 2, the analytic solution is presented for a comparatively large value $c = 5.0$ and $F_0 = 1.0$. Since time is incorporated in the spatial similarity variables, the figures display the solutions at all times. In this case, the initial condition is already displayed in figures 1 and 2 when we assume t to be zero. Compared to the initially singular solutions of Philip and Knight (1991), our solutions have the additional advantage of being finite at $t = 0$. As can be seen from figures 1 and 2, the initial condition is an almost symmetric distribution of material which could have resulted from a local injection. The

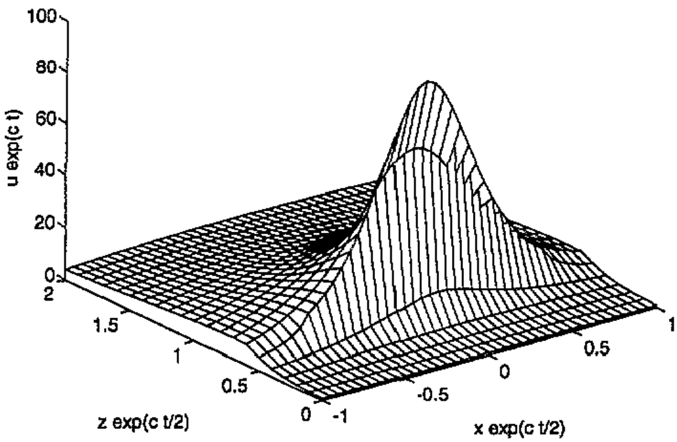

Figure 3. Plot of (17) for $c = 1.0$ and $F_0 = 1.0$

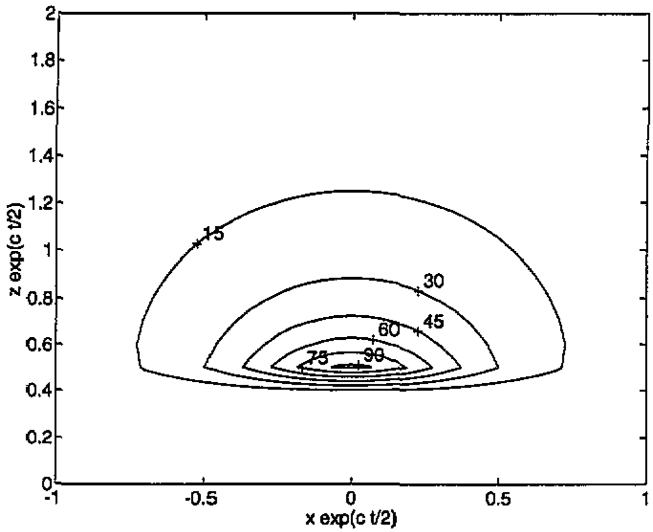

Figure 4. Contour plot of (17) for $c = 1.0$ and $F_0 = 1.0$.

injected slug is allowed to spread, without extra material being supplied at the origin.

In figures 3 and 4, the analytic solution is presented for a comparatively small value, $c = 1.0$ and $F_0 = 1.0$. Again, the initial condition is evident when $t = 0$. The concentration peaks in figures 3 and 4 are steeper than in figures 1 and 2. The initial condition could have originated from an injection from a horizontal line segment source.

We note that the solutions are symmetric about the plane $x = 0$. As c becomes larger, the solution reduces to a one-dimensional steady state. The one-dimensional steady-state solution to (1), which has been used to model evaporation from a soil with a water table, can be obtained for arbitrary $D(u)$ and $K(u)$ (Gardner 1958).

4. The 3D nonlinear diffusion–convection equation

For classical Lie group analysis of the class of equations (1) in three dimensions, we are again interested in the general equation, so that as in the 2D analysis, $D(u)$ and $K(u)$ are arbitrary. Through symmetry analysis, we identify the special forms of D and K which possess extra symmetries.

The infinitesimal transformations appropiate in this instance are

$$\begin{aligned}
u_* &= \mathrm{e}^{\epsilon\Gamma}u = u + \epsilon\,\mathcal{U}(x,y,z,t,u) + \mathrm{O}(\epsilon^2)\\
t_* &= \mathrm{e}^{\epsilon\Gamma}t = t + \epsilon\,\mathcal{T}(x,y,z,t,u) + \mathrm{O}(\epsilon^2)\\
x_* &= \mathrm{e}^{\epsilon\Gamma}x = x + \epsilon\,\mathcal{X}(x,y,z,t,u) + \mathrm{O}(\epsilon^2)\\
y_* &= \mathrm{e}^{\epsilon\Gamma}y = y + \epsilon\,\mathcal{Y}(x,y,z,t,u) + \mathrm{O}(\epsilon^2)\\
z_* &= e^{\epsilon\Gamma}z = z + \epsilon\,\mathcal{Z}(x,y,z,t,u) + O(\epsilon^2)
\end{aligned} \tag{19}$$

where the infinitesimal generator Γ (Olver 1986, Bluman and Kumei 1989) is

$$\Gamma = \mathcal{X}\frac{\partial}{\partial x} + \mathcal{Y}\frac{\partial}{\partial y} + \mathcal{Z}\frac{\partial}{\partial z} + \mathcal{T}\frac{\partial}{\partial t} + \mathcal{U}\frac{\partial}{\partial u}\,.$$

Equation (19) is then extended to first and second order, so that, for example,

$$\frac{\partial u_*}{\partial x_*} = \frac{\partial u}{\partial x} + \epsilon\,\mathcal{U}_1 + \mathrm{O}(\epsilon^2)$$

where

$$\mathcal{U}_1 = \frac{\mathrm{D}}{\mathrm{D}x}\mathcal{U} - \left(\frac{\mathrm{D}}{\mathrm{D}x}\mathcal{X}\right)u_x - \left(\frac{\mathrm{D}}{\mathrm{D}x}\mathcal{Y}\right)u_y - \left(\frac{\mathrm{D}}{\mathrm{D}x}\mathcal{Z}\right)u_z - \left(\frac{\mathrm{D}}{\mathrm{D}x}\mathcal{T}\right)u_t \tag{20}$$

with $\mathrm{D}/\mathrm{D}x$ *the total derivative operator with respect to* x,

$$\frac{\mathrm{D}}{\mathrm{D}x}\,F(x,y,z,t,u) = \frac{\partial F}{\partial x} + u_x\frac{\partial F}{\partial u}\,.$$

For arbitrary D and K, the symmetries are

$$\Gamma_1 = \frac{\partial}{\partial x} \qquad \Gamma_2 = \frac{\partial}{\partial y} \qquad \Gamma_3 = \frac{\partial}{\partial z} \qquad \Gamma_4 = \frac{\partial}{\partial t} \qquad \Gamma_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

where Γ_1 to Γ_4 are the generators of space and time translations, and Γ_5 is the generator of rotations about the vertical axis. As in the 2D case, the only forms of $D(u)$, up to a linear change of variables, which have additional symmetries are power law and exponential. In table 2, Γ_1 to Γ_5 are omitted, as they are common to all cases. The case $K'(u)$ constant has not been included as this form of (1) can be transformed to the case of pure diffusion, fully analysed by Galaktionov *et al* (1986).

The three-dimensional Burgers' equation, like the 2D Burgers' equation, is the case which possesses the most symmetries. The symmetries Γ_6 and Γ_7 are compatible, as they obey the commutation property $[\Gamma_6, \Gamma_7] = \Gamma_7$. We utilize the symmetries in table 2 to *obtain an exact solution to a 3D diffusion–convection equation.*

Equation (1) is transformed from Cartesian space coordinates (x, y, z) into spherical coordinates (r, γ, ψ) using

$$x = r\sin\gamma\cos\psi \qquad y = r\sin\gamma\sin\psi \qquad z = r\cos\gamma$$

Table 2. Symmetries for the 3D diffusion–convection equation $m, n \in \Re$.

$D(u)$	$K'(u)$	Γ_i
u^m	u^n	$\Gamma_6 = (m-2n)t\frac{\partial}{\partial t} + (m-n)x\frac{\partial}{\partial x} + (m-n)y\frac{\partial}{\partial y} + (m-n)z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}$
u^m	$\ln(u)$	$\Gamma_6 = mt\frac{\partial}{\partial t} + mx\frac{\partial}{\partial x} + my\frac{\partial}{\partial y} + (mz+t)\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}$
e^{mu}	e^{nu}	$\Gamma_6 = (m-2n)t\frac{\partial}{\partial t} + (m-n)x\frac{\partial}{\partial x} + (m-n)y\frac{\partial}{\partial y} + (m-n)z\frac{\partial}{\partial z} + \frac{\partial}{\partial u}$
e^{mu}	u	$\Gamma_6 = mt\frac{\partial}{\partial t} + mx\frac{\partial}{\partial x} + my\frac{\partial}{\partial y} + (mz+t)\frac{\partial}{\partial z} + \frac{\partial}{\partial u}$
const	u	$\Gamma_6 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}$
		$\Gamma_7 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial u}$

yielding the nonlinear PDE

$$\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 D(u)\frac{\partial u}{\partial r}\right] + \frac{1}{r^2 \sin\gamma}\frac{\partial}{\partial \gamma}\left[\sin\gamma\, D(u)\frac{\partial u}{\partial \gamma}\right] + \frac{1}{r^2\sin^2\gamma}\frac{\partial}{\partial \gamma}\left[D(u)\frac{\partial u}{\partial \psi}\right]$$
$$-K'(u)\left[\cos\gamma\frac{\partial u}{\partial r} - \frac{\sin\gamma}{r}\frac{\partial u}{\partial \gamma}\right]. \tag{21}$$

Assuming power-law functions for D and K' (that is, $D(u) = u^m, K'(u) = u^n$), the symmetries from table 2 become

$$\begin{aligned}
\Gamma_1 &= \sin\gamma\cos\psi\frac{\partial}{\partial r} + \frac{\cos\gamma\cos\psi}{r}\frac{\partial}{\partial \gamma} - \frac{\sin\psi}{r\sin\gamma}\frac{\partial}{\partial \psi} \\
\Gamma_2 &= \sin\gamma\sin\psi\frac{\partial}{\partial r} + \frac{\cos\gamma\sin\psi}{r}\frac{\partial}{\partial \gamma} + \frac{\cos\psi}{r\sin\gamma}\frac{\partial}{\partial \psi} \\
\Gamma_3 &= \cos\gamma\frac{\partial}{\partial r} - \frac{\sin\gamma}{r}\frac{\partial}{\partial \gamma} \\
\Gamma_4 &= \frac{\partial}{\partial t} \\
\Gamma_5 &= \frac{\partial}{\partial \psi} \\
\Gamma_6 &= (m-2n)t\frac{\partial}{\partial t} + (m-n)r\frac{\partial}{\partial r} + u\frac{\partial}{\partial u}.
\end{aligned} \tag{22}$$

Philip and Knight (1991) assume the functional form

$$u = F(\rho, \gamma, \psi)\, t^{-\alpha} \qquad \rho = r t^{-\alpha/3} \tag{23}$$

which ensures total material conservation. The values of α and n are shown to be

$$\alpha = \frac{3}{3m+2} \qquad n = m + \tfrac{1}{3} \qquad m \neq -\tfrac{2}{3}.$$

However, from Γ_6 (when $n \neq m$), solving the characteristic equation

$$\frac{\mathrm{d}r}{(m-n)r} = \frac{\mathrm{d}\gamma}{0} = \frac{\mathrm{d}\psi}{0} = \frac{\mathrm{d}t}{(m-2n)t} = \frac{\mathrm{d}u}{u}$$

leads to the functional form

$$u = F(\rho, \gamma, \psi)\, t^{-1/(2n-m)} \qquad \rho = r t^{(m-n)/(2n-m)} \tag{24}$$

which will lead to a reduction of order of the PDE for general m and n, ($n \neq m$). If we demand total conservation of material, we then obtain the form

$$u = F(\rho, \gamma, \psi)\, t^{-3/(3m+2)} \qquad \rho = r t^{-1/(3m+2)} \qquad m \neq -\tfrac{2}{3} \tag{25}$$

which is the same as that obtained by Philip and Knight (1991). We see that the case $m = -\frac{2}{3}$ is not included by this reduction. In this case, Γ_6 is replaced by

$$\Gamma_6 = r\frac{\partial}{\partial r} - 3u\frac{\partial}{\partial u}$$

and taking a linear combination of Γ_6 and Γ_7, the characteristic equation is

$$\frac{\mathrm{d}r}{r} = \frac{\mathrm{d}\gamma}{0} = \frac{\mathrm{d}\psi}{0} = \frac{\mathrm{d}t}{\beta} = \frac{\mathrm{d}u}{-3u}$$

with $\beta \in \Re$, $\beta \neq 0$. Letting $c = 3/\beta$ and solving leads to the functional form

$$u = F(\rho, \gamma, \psi)\,\mathrm{e}^{-ct} \qquad \rho = r\mathrm{e}^{-ct/3}. \tag{26}$$

We note that (26) always ensures total material conservation, and will lead to a reduction by one in the number of independent variables. As in the 2D special case, the arbitrary constant c leads to greater variability in the solution.

Total material conservation requires zero flow at the origin:

$$\text{for}\quad t > 0 \quad r = 0 \qquad \frac{\partial u}{\partial r} = \tfrac{3}{2}\cos\gamma\, u^{4/3}. \tag{27}$$

Substitution of (26) into (21) gives

$$\frac{-c}{3\rho^2}\frac{\partial}{\partial\rho}\left[\rho^3 F\right] = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left[\rho^2 F^{-2/3}\frac{\partial F}{\partial\rho}\right] + \frac{1}{\rho^2\sin\gamma}\frac{\partial}{\partial\gamma}\left[\sin\gamma\, F^{-2/3}\frac{\partial F}{\partial\gamma}\right]$$
$$+\frac{1}{\rho^2\sin^2\gamma}\frac{\partial}{\partial\gamma}\left[F^{-2/3}\frac{\partial F}{\partial\psi}\right] - F^{-1/3}\left[\cos\gamma\frac{\partial F}{\partial\rho} - \frac{\sin\gamma}{\rho}\frac{\partial F}{\partial\gamma}\right] \tag{28}$$

while the zero flux condition (27) becomes

$$\text{for}\quad t > 0 \qquad \rho = 0 \qquad \frac{\partial F}{\partial\rho} = \tfrac{3}{2}\cos\gamma\, F^{4/3}. \tag{29}$$

The similarity reduction (26) implies that the flux has no component normal to the radii, as in the case of similarity variables with power-law dependence (Philip and Knight 1991). This means that we have the two additional constraints

$$\frac{-1}{\rho}F^{-2/3}\frac{\partial F}{\partial\gamma} - \tfrac{3}{2}\sin\gamma\, F^{2/3} = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial\psi} = 0. \tag{30}$$

Utilizing these constraints reduces (28) to the ODE

$$-\tfrac{1}{3}c\rho F = F^{-2/3}\frac{\partial F}{\partial\rho} - \tfrac{3}{2}\cos\gamma\, F^{2/3}. \tag{31}$$

This may be solved exactly (Kamke 1959) to give the solution ($c > 0$)

$$\left(\frac{\eta + 2\rho\eta^{1/2}\cos\gamma - \frac{4}{9}c\rho^2}{\eta_0}\right) = \left(\frac{\eta^{1/2} + \rho(\cos\gamma - \sqrt{\cos^2\gamma + 4c/9})}{\eta^{1/2} + \rho(\cos\gamma + \sqrt{\cos^2\gamma + 4c/9})}\right)^{\cos\gamma/2\sqrt{\cos^2\gamma+4c/9}} \tag{32}$$

where $\eta = 1/F$. The case $c < 0$ has been neglected to avoid singularities.

Since this is axially symmetric, solutions are similar in appearance to those obtained in two dimensions, except that Cartesian coordinate x is replaced by $(x^2 + y^2)^{1/2}$.

Closed-form solutions have previously been obtained using the reduction method of Philip and Knight (1991) only in the cases $(m, n) = \left(0, \frac{1}{3}\right)$ and $(m, n) = \left(\frac{1}{3}, \frac{2}{3}\right)$.

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